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THE SPECIAL FUNCTIONS OF MATHEMATICS AND ITS APPLICATIONS

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ABSTRACT

New generalized Lauricella functions and many integral theorems on the generalized Lauricella function are given and defined here utilizing the Riemann-Liouville fractional integral and differential operators. This section focuses on the expanded second Appell hyper geometric generating function introduced by Parmar et al. in their article obtained numerous well-known conclusions as special cases of several previously reported features on hyper geometric functions, including integration and differentiation, with the introduction of the generalized k-Fractional derivative. Understanding the relationship between the k-Gauss and k-Confluent hyper geometric functions may be accomplished through the use of the Laplace transform. The generalized k-Fractional derivative was used to derive the power function image, Laplace transform, and k-Riemann Liouville fractional integral.

Keywords: *Special Functions, Mathematics, Applications, and Hyper geometric Function.*

INTRODUCTION

John Wallis, an Oxford professor in the seventeenth century, advanced the notion of the Gamma function and provided the formula for. Aside from Wallis, Cavalieri was also responsible for the elliptic integrals; In the seventeenth century, two types of particular roles were encountered. For most of the eighteenth century, Wallis elliptic integral's hyper geometric functions and Newton's basic symmetric functions remained a mystery. Newton and Leibnitz developed calculus in Cambridge and Leibnitz, respectively. in Germany between 1665 and 1685 is responsible for the real projections and elevations of special functions.[1]

Gregory discovered Taylor's theorem in 1670, but it wasn't published until 1715 when Taylor rediscovered it. As a solution to the differential equation by an infinite series, James Bernouli derived a series representation of a Bessel's function. Euler's study in 1730 assessed the majority of the gamma function's key characteristics and the beta function integral was calculated in Euler's work in 1772, both in terms of the gamma function. Furthermore, Vandermonde's theory was also created in 1772. The nineteenth century has been dubbed the golden era of special functions because of the numerous uses of special functions in both mathematics and science.

In the history of special functions, C.F. Gauss's theory of hyper geometric series 2 F 1 from 1812 constituted a turning point. Kummer launched the 1F1 series in 1830, a year after Clausen finished work on the 3F2 series. First introduced by Appell in 1880, Lauricella improved the hyper geometric functions to accommodate additional variables in 1893. There were various developments of 2 F 1 based on Barnes's gamma function, such as Horn,Kampe de Feriet, Macrobert, and Meijer's expansions of 2 F 1. The groundwork for the theory of special functions, Redefining it is an ongoing process as new successes and ideas emerge in related fields like as geometry, group theory, partition theory, number theory, and so on Special functions have been used to examine a wide range of powerful instruments in a variety of physical areas.[3]

GAUSSIAN HYPERGEOMETRIC FUNCTION

The Pochhammer sign, the Gamma function, and associated functions are used to denote the Gauss hyper geometric series and its generalization.

Gamma Function

Simplest and most essential special function is gamma. The majority of Euler's definitions are equivalent.

We follow Euler in defining the function $\Gamma(z)$, by

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt, \qquad z\in C, \mathrm{Re}(z)>0$$

Gamma function and Pochhammer's symbol are

The Pochhammer's symbol $(\lambda)_n$ is defined as[4]

$$(\lambda)_n=\begin{cases}1 & if\ n=0\\ \lambda(\lambda+1)(\lambda+2)......(\lambda+n-1) & if\ n=1,2,3,....\end{cases}$$

where $;(1)_n=n!,\ (\lambda)_n$ basic factorials may be seen as a generalization Using the Gamma function, the Pochhammer's symbol may be defined as

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \qquad \lambda\neq 0,-1,-2,......$$

And $(\lambda)_{m+n} = (\lambda)_m (\lambda+m)_n$

Beta function

The Beta function $B(m,n)$ is defined as[5]

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt,\ \mathrm{Re}(m) > 0, \mathrm{Re}(n) > 0$$

There is interplay between Gamma and Beta functions.

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ \mathrm{Re}(m) > 0, \mathrm{Re}(n) > 0$$

The Gaussian Hyper geometric Series

Formulated by the German mathematician C.F.Gauss (1812)

$$\sum_{n=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} = 1 + \frac{a.b}{1.c} z + \frac{a.(a+1)b.(b+1}{1.2.c.(c+1)} z^2 + \ldots\ldots\ldots\ldots\ldots$$

$$= {}_2F_1(a,b;c;z), a,b,c \in C, c \neq 0,-1,-2,\ldots$$

Where (2F1) a,b;c; z is known as Gauss hyper geometric function. The hyper geometric series Converge completely around the unit sphere $|z| < 1,$ Re(c − a − b) > 0 for z = 1. The Equation is reduces in the elementary geometric series when a = c and b =1or a =1 and b = c ,

$$\sum_{k=0}^{\infty} z^k = 1 + z + z^2 + \ldots\ldots\ldots\ldots$$

There is no Gauss hyper geometric function if one of the parameters aorb is negative integer, and if one of the variable c is 0, the equation is undefined $-c < -a$. In the equation, if we replace z by z/b and taking limit $b \to \infty$ then,

We get a well known function called Kummer's function

$${}_1F_1(a;c;z) = 1 + \frac{a.}{1.c} z + \frac{a.(a+1)}{1.2.c.(c+1)} z^2 + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!} \qquad \{|z| < 1, c \neq 0,-1,-2\ldots\}$$

The confluent hyper geometric function is another name for it.

Confluent hyper geometric functions contain special instances like parabolic cyllindric functions and coulomb wave functions, while the hyper geometric function 2 F 1 includes specific situations like first and second class incomplete elliptic functions, incomplete beta functions, and Bessel functions. Many eminent mathematicians like C.F.Gauss, R. Mellin, Y.L. Luke, E.W. Barnes, E.E.Kummer have explored the function 2 *F* 1 and 1 *F1*. The hyper geometric function has further generalized by number of eminent mathematician.[6]

GENERALIZED HYPERGEOMETRIC FUNCTION

The generalized hyper geometric function *pFq*, is a function 2F1 is a straightforward generalization of the hyper geometric function

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = {}_pF_q\left[\begin{matrix} a_1,\ldots a_p \\ \\ b_1,\ldots b_q \end{matrix} ;z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!} $$

There are two factors in this equation: the numerator and the denominator, which are p and q. They are either positive integers or zero in this instance. If the denominator is neither 0 or a negative integer, a complex number may be used. in a series.[7]

On the circle z =1and $p = q + 1$, if we set

$$ w = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j $$

The series is

(i) Absolutely convergent if $\mathrm{Re}(w) > 0$

(ii) Conditionally convergent if $-1 < \mathrm{Re}(w) \leq 0$ for $z \neq 1$

(iii) Divergent for $\mathrm{Re}(w) \leq -1$

An attempt was made at providing some context for P F q, $p > q + 1$, MacRobert defined the E- function. Meijer used a Mellin-Barnes type counter integral to further extend the E function, and he examined the characteristics of the newly established G-function in a series of papers from 1946 through 1956. Luke's study also provides a full explanation of the G-function. The significance of the G-function in statistical distributions has been emphasized by Mathai and Saxena. In 1961, Charles Fox created a more generic function known as Fox's H-function, which is commonly referred to in literature.[8]

E.M. Wright, generalize the *p F q* in the following form:

$$
{}_p\psi_q\left[\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix};x\right]={}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\ \\ (b_1,\beta_1)_{1,q}\end{matrix};x\right]
$$

$$
=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}
$$

Here αi and β*j* are real and positive quantities and *i* = 1, ..., *p* , *j* = 1, ..., *q* and

$$
1+\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>0
$$

When αi and β*j* are equal to 1, the only difference between this equation and the generalized hyper geometric function is the multiplier, which is a constant. Dotsenko and Malovichko studied the generalized hyper geometric function, and one of Dotsenko's particular examples was this as,[9]

$$
{}_2R_1^{\omega,\mu}(z)={}_2R_1(a,b;c;\omega,\mu;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)}\frac{z^n}{n!}
$$

In 2001, Virenchenko et al. developed this hyper geometric function of the Wright Type by taking $\frac{\omega}{\mu}=\tau>0$ in equation.

$$
{}_2R_1^{\tau}(z)={}_2R_1(a,b;c;\tau;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}
$$

An inclusive version of 2 *F* 1, 1 *F* 1 and *pFq* functions can be set up in the standard works by Exton, Luke, Rainville and Slater.[10]

HYPERGEOMETRICFUNCTIONS OF TWOVARIABLES

Following this success in one variable, researchers investigated and produced a two-variable version of the hyper geometric function. Four double hyper geometric series were developed by P.Appell in 1880:

$$F_1(a,b,b';c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b_m)(b')_n}{(c)_{m+n}}\frac{(x)^m}{m!}\frac{(y)^n}{n!},\quad (|x|<1,|y|<1).$$

$$F_2(a,b,b';c,c';x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n}\frac{x^m y^n}{m!n!},\quad (|x|+|y|<1).$$

$$F_3(a,a',b,b';c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}}\frac{x^m y^n}{m!n!},$$

$$\max\{|x|,|y|\}<1.$$

$$F_4(a,b;c,c';x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n}\frac{x^m y^n}{m!n!},\quad \left(\sqrt{|x|}+\sqrt{|y|}<1\right).$$

As is customary, neither the numerator nor the denominator parameters c nor c' are a negative integer and $F1$, $F2$, $F3$ and $F4$ are the generalization of Gaussian Hyper geometric function. Horn added 10 additional series (G1, G2, G3, H1,...,H7) to complete the Appell function set. Humbert explored confluent versions of these functions in Erdelyi et alstudy, .'s which included the full list of functions. Due to the usefulness of the theory of hyper geometric function for one and two variables, many mathematicians investigated and created an analogous theory for hyper geometric functions with more than two complex variables. It was Lauricella who, in 1893, extended the four functions (F1, F2, F3, and F4) to numerous hyper geometric functions, giving rise to the phrase "Lauricella function.[11]

$$F_A^{(n)}\left[a,b_1,\dots b_n;c_1,\dots c_n;x_1,\dots x_n\right]=$$
$$\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a)_{m_1+\dots+m_n}(b_1)_{m_1}\dots\dots(b_n)_{m_n}}{(c_1)_{m_1}\dots(c_n)_{m_n}}\frac{(x_1)^{m_1}}{m_1!}\dots\frac{(x_n)^{m_n}}{m_n!},$$
$$\{|x_1|+\dots|x_n|<1\}.$$

$$F_B^{(n)}\left[a_1,\dots,a_n,b_1,\dots b_n;c;x_1,\dots x_n\right]=$$
$$\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a_1)_{m_1}\dots(a_n)_{m_n}(b_1)_{m_1}\dots\dots(b_n)_{m_n}}{(c)_{m_1+\dots+m_n}}\frac{(x_1)^{m_1}}{m_1!}\dots\frac{(x_n)^{m_n}}{m_n!},$$
$$\max\{|x_1|,\dots,|x_n|<1\}.$$

$$F_C^{(n)}\left[a,b;c_1,\dots c_n;x_1,\dots x_n\right]=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a)_{m_1+\dots+m_n}(b)_{m_1+\dots+m_n}}{(c_1)_{m_1}\dots(c_n)_{m_n}}\frac{(x_1)^{m_1}}{m_1!}\dots\frac{(x_n)^{m_n}}{m_n!},$$

$$\{\sqrt{|x_1|}+\dots+\sqrt{|x_n|}<1\}.$$

$$F_D^{(n)}\left[a,b_1,\dots b_n;c;x_1,\dots x_n\right]=$$
$$\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a)_{m_1+\dots+m_n}(b_1)_{m_1}\dots(b_n)_{m_n}}{(c)_{m_1+\dots+m_n}}\frac{(x_1)^{m_1}}{m_1!}\dots\frac{(x_n)^{m_n}}{m_n!},$$
$$\max\{|x_1|,\dots,|x_n|<1\}.$$

Several authors, including Humbert, Erdelyi, and Exton, have addressed limited examples of Lauricella's function. Many writers have examined the extension of the Lauricella functions and certain of its confluent forms.

INTEGRAL TRANSFORM

For solving differential and integral equations in applied mathematics, integral transformations using special functions are also critical. If $K(x,s)$ As is customary, neither the numerator nor the denominator parameters c nor c' are a negative integer x and s, on the interval $a \le x \le b$, next comes the function's transformation $f(x)$ with respect to $K(x,s)$ called the kernel is defined by,

$$T\left[f(x);s\right]=\int_0^{\infty}K(x,s)f(x)\,dx\ \dots$$

Where $f(x)$ the real variable t must be a real or complex valued function

Also if $f(x)$ it is possible to say $T\left[f(x);s\right]$ by a form-integral component

$$f(x)=\int_{\alpha}^{\beta}T\left[f(x);s\right]\theta(x,s)\,ds\ \dots$$

Equation inversion formula is the name given to this type of equation.

Laplace Transform

As a result of its widespread use in practical mathematics and physics, Laplace's transform is one of the most significant and simplest variants of the integral transform. System analysis and differential equations may be solved with this tool, the Laplace transform is the most powerful mathematical approach available.[12]

$$L\left[f(x);s\right]=\bar{f}(s)=\int_0^{\infty}e^{-sx}f(x)\,dx$$

The inversion formula for the above integral can be found if it exists.

$$L^{-1}\left[\bar{f}(s)\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx} f(x)\,ds$$

The Laplace transform has been generalized by a variety of writers.

Mellin Transform

Following are the rules for defining the well-known Mellin[13] transform and it's inverse.

$$M\left[f(x);s\right]=\bar{f}(s)=\int_{0}^{\infty} x^{s-1} f(x)\,dx$$

And its inversion formula is given by

$$M^{-1}\left[\bar{f}(s)\right]=f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} f(s)\,ds$$

(Euler) Beta Transforms

Euler's beta transformations are defined as follows:

$$B\left[f(x);a,b\right]=\int_{0}^{1} x^{a-1}(1-x)^{b-1} f(x)\,dx$$

Whittaker Transforms

It is defined as the Whittaker function[14]

$$\int_{0}^{\infty} e^{-x/2}\, x^{v-1}\, W_{\lambda,\mu}(x)\,dx=\frac{\Gamma\left(\frac{1}{2}+\mu+v\right)\Gamma\left(\frac{1}{2}-\mu+v\right)}{\Gamma(1-\lambda+v)},$$

Where $\mathrm{Re}(\mu\pm v)>-\frac{1}{2}$ and $W_{\lambda,\mu}(x)$ is the Whittaker confluent hyper geometric function[15].

CONCLUSION

The developed integral formulae extend and unify a significant number of previously published findings, therefore expanding the range of possible applications. Using the product of the I-function and the general class of polynomials, the pictures of the generalized fractional integral operators provided by Saigo have been improved further. Aside from the fact that these conclusions are quite general, they have been presented in a compact way, eliminating endless series and so making them helpful in applications. The approximated the

CDF for this distribution using Chèbyshev polynomials and the error function. The CDF of probability distributions can be approximated using the methods we used since their CDF is limited and takes values from 0 to 1. The best polynomials for approximating continuous functions are Chèbyshev polynomials, as is well-known. However, rational Chèbyshev approximants can also be used to approximate such functions.

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